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SUM AND DIFFERENCE SETS IN GENERALIZED QUATERNION GROUPS

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Abstract. Given a group G, we say that a set $A \subseteq G$ has more sums than differences (MSTD) if $|A+A| > |A-A|$, has more differences than sums (MDTS) if $|A+A| < |A-A|$, or is balanced if $|A+A| = |A-A|$. A problem of recent interest has been to understand the frequencies of these types of subsets. It is known that for arbitrary finite groups G, almost all subsets $A \subseteq G$ are balanced sets as $|G| \to \infty$. Recently for the generalized dihedral groups $D = \mathbb{Z}_2 \ltimes G$, it is conjectured that there are more MSTD sets than MDTS sets. In this paper, we investigate the behavior of the sum and difference sets of $A \subseteq Q_{4n}$, where Q_{4n} denotes generalized quaternion groups and show that the generalized quaternion group Q_{4n} has at least 2^{2n} subsets which are MSTD. We also analyze the expectation for $|A - A|$ where $A \subseteq Q_{4n}$, proving an explicit formula for $|A - A|$ when n is prime.

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1. Introduction:

If $A \subseteq \mathbb{Z}$, the sumset and difference set of A are defined respectively as,

$$A + A := \{x \in \mathbb{Z} : x = a_1 + a_2 \text{ for some } a_1, a_2 \in A\},$$

$$A - A := \{x \in \mathbb{Z} : x = a_1 - a_2 \text{ for some } a_1, a_2 \in A\}.$$

These elementary operations are fundamental in additive number theory. A natural problem of recent interest has been to understand the relative sizes of the sum and difference sets of a set of integers A.

Definition 1.1. *If* $|A + A| > |A - A|$*, we say* A *is a* More Sums Than Differences (MSTD) *set or a* sum-dominated *set , while if* $|A+A| = |A-A|$ *we say* A *is* balanced*, and if* $|A + A| < |A - A|$ *then* A *is a More Differences Than Sums(MDTS) set or a* difference-dominated *set.*

Generally, we expect most sets to be MDTS since addition is commutative and subtraction is not. Nevertheless, MSTD subsets of integers exist. Nathanson detailed in [6] the history of the problem and attributed to John Conway the first recorded example of an MSTD subset of integers, $\{0, 2, 3, 4, 7, 11, 12, 14\}$. Martin and O'Bryant proved in [4] that the proportion of the 2^n subsets A of $\{0, 1, \dots, n-1\}$, which are MSTD, is bounded below by a positive value for all $n \geq 15$. They proved this by controlling the "fringe" elements of A, those close to 0 and $n-1$, which have the most influence over whether elements are missing from the sum and difference sets. In [10], Zhao gave a deterministic algorithm to compute the proportion of

MSTD subsets of $\{0, 1, \ldots, n-1\}$ as n goes to infinity and found that this proportion is at least 4.28×10^{-4}.

Much of the study of sum-dominant sets has concerned subsets of the integers. However, the phenomenon in finite abelian groups has received some attention, notably in [3, 6, 9]. In [7], Nathanson showed that families of MSTD sets of integers could be constructed from MSTD sets in finite abelian groups.

For finite groups, although the usual notation for the operation of the group is multiplication, we match the notation from previous work and define, for a subset $A \subseteq G$, its sumset and difference set as

$$A + A = \{a_1 a_2 : a_1, a_2 \in A\},$$

and

$$A - A = \{a_1 a_2^{-1} : a_1, a_2 \in A\}.$$

Martin and O'Bryant showed in [4] that although MSTD subsets of the integers are rare, they are a positive percentage of subsets of $\{0, 1, \ldots, n-1\}$. MSTD sets in finite groups are even rarer. In [5], Miller and Vissuet proved that as the size of a finite group tends to infinity, the probability that a subset chosen uniformly at random is sum-dominant tends to zero. Somewhat surprisingly, this is also true for difference-dominant sets. This is very different from the integer case, where more than 99.99% of all subsets are difference-dominant.

The reason integers behave differently from finite groups is that a subset of integers contains fringe elements, as noted earlier.

Let S be a subset of $I_n := \{0, 1, \ldots, n\}$ chosen uniformly at random. The elements of S near 0 and n are the fringe elements. Interestingly the notion of nearness is independent of n, and the reason is that almost all possible elements of $I_n + I_n$ and $I_n - I_n$ are realized respectively by $S + S$ and $S - S$.

Thus, whether or not a set is sum-dominant is essentially controlled by the fringe elements of S, as the 'middle' is filled with probability one, and the presence and absence of fringe elements control the extremes. In a finite group, there are no fringe elements since each element can be written as $|G|$ different sums and differences, and thus most elements appear in the sumset or difference set with high probability.

2. Some Important Results

Theorem 2.1 ([5]). *Let $\{G_n\}$ be a family of finite groups, not necessarily abelian, such that $|G_n| \to \infty$. If S_n is a uniformly chosen random subset of G_n, then the probability $\mathbb{P}(S_n + S_n = S_n - S_n = G_n) \to 1$ as $n \to \infty$. In other words, as the size of the finite group grows, almost all subsets are balanced (with sumset and difference set being equal to the entire group).*

Miller and Vissuet, who studied the dihedral groups in [5], conjectured that for $n \geq 3$, D_{2n} has more MSTD subsets than MDTS subsets. Recently, Ascoli et al. in [1] made progress towards this conjecture by partitioning subsets of D_{2n} by their size. They conjectured the following.

Conjecture 2.2 ([1]). *Let G be an abelian group with at least one element of order 3 or greater, and let $D = \mathbb{Z}_2 \ltimes G$ be the corresponding generalized dihedral group. Then, there are more MSTD subsets of D than MDTS subsets of D.*

Theorem 2.3 ([1])**.** *Let $D = \mathbb{Z}_2 \ltimes G$ be a generalized dihedral group of size $2n$. Let $S_{D,m}$ denote the collection of all subsets of D of size m, and let j denote the number of elements in G with the order at most 2. If $6 \le m \le c_j\sqrt{n}$, where $c_j = 1.3229\sqrt{111 + 5j}$, then there are more MSTD sets than MDTS sets in $S_{D,m}$.*

In this paper, we will explore the sumset and difference set of the quaternion group with the help of results based on the dihedral group.

3. Quaternion Group

The quaternion group Q_8 (sometimes just denoted by Q) is a non-abelian group of order eight, isomorphic to the eight-element subset $\{1, i, j, k, -1, -i, -j, -k\}$ of the quaternions under multiplication. It is given by the group presentation,
$\mathrm{Q}_8 = \langle -1, i, j, k \mid (-1)^2 = 1,\ i^2 = j^2 = k^2 = ijk = -1\rangle$.

3.1. Generalised Quaternion Group.

Definition 3.1. *The generalized quaternion group of order $4n$ is defined as*

$$\mathrm{Q}_{4n} = \langle a, b \mid a^n = b^2, a^{2n} = b^4 = 1, b^{-1}ab = a^{-1}\rangle,$$

where $n \ge 2$.

With this definition, we can consider the specific example of the quaternion group of order eight. The restrictions on generators for Q_8 are as follows:

$$\mathrm{Q}_8 = \langle a, b \mid a^2 = b^2, b^4 = 1, b^{-1}ab = a^{-1}\rangle.$$

We can write the eight elements in the following form: $1, a, a^2, a^3, b, ab, a^2b, a^3b$. Generally, one insists that $n > 1$ as the properties of generalized quaternions become more uniform at this stage. However if $n = 1$ then one observes $a = b^2$, so $Q_4 \cong \mathbb{Z}_4$. Dihedral group properties are strongly related to generalized quaternion group properties because of their highly related presentations. We will see this in many of our results.

The following result is well-known in [8].

Proposition 3.2 ([8])**.** *The elements $\{1, a, a^2, \ldots, a^{2n-1}, b, ab, a^2b, \ldots, a^{2n-1}b\}$ represent the $4n$ distinct group elements of Q_{4n}.*

. Below, we list the sumsets and difference sets in the Quaternion Groups for $n = 1$ and $n = 2$.

- If $n = 1$, $Q_4 = \{1, -1, i, -i\}$, all subsets of Q_4 are balanced.
- If $n = 2$, we have $\mathrm{Q}_8 = \langle -1, i, j, k \mid (-1)^2 = 1,\ i^2 = j^2 = k^2 = ijk = -1\rangle$.
  1. If we take a subset of cardinality 1, then both sumset and difference set have also cardinality 1, therefore balanced.
  2. We can write Q_8 as $\{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Let $A = \{a_1, a_2\}$ be a subset of Q_8 of cardinality 2 where $a_1, a_2 \in Q_8$.
     $A + A = \{a_1^2, a_1a_2, a_2a_1, a_2^2\}$ and $A - A = \{1, a_1a_2^{-1}, a_2a_1^{-1}\}$. Since $a_1^2 = a_2^2 = -1$ except a_1 or a_2 belongs to $\{-1, 1\}$ and if either $a_1 = \pm 1$ or $a_2 = \pm 1$ then $a_1a_2 = a_2a_1$ therefore in every case $|A + A| = |A - A|$.
  3. Let $A = \{a_1, a_2, a_3\}$, where $a_1, a_2, a_3 \in Q_8$ then
     $A + A = \{a_1^2, a_1a_2, a_1a_3, a_2a_1, a_2a_3, a_2^2, a_3a_1, a_3a_2, a_3^2\}$ and
     $A - A = \{1, a_1a_2^{-1}, a_1a_3^{-1}, a_2a_1^{-1}, a_2a_3^{-1}, a_3a_1^{-1}, a_3a_2^{-1}\}$, where $a_1^2 = a_2^2 = a_3^2 = -1$. If any two of a_1, a_2, a_3 are in $\{\pm i, \pm j, \pm k\}$ and the other is ± 1,

then $a_i s$ will commute with each other. So in every case, the cardinality of $A+A$ is less than or equal to the cardinality of $A-A$, so we will not get any MSTD set but if $A=\{i,j,1\}$ then $|A+A|=6$ and $|A-A|=7$, therefore, we have MDTS sets but no MSTD sets. In fact, we have 24 MDTS sets in Q_8 of cardinality 3.

(4) Let $|A|=4$ and $A=\{a_1,a_2,a_3,a_4\}$. The number of such subsets is $\binom{8}{4}=70$. Consider the following cases.
   - Case 1: Fixing two elements as 1 and -1, for the remaining two elements, there are $\binom{6}{2}=15$ choices; hence in total, there are 15 different sets of this type, and in each case, we found $|A-A|\geq|A+A|$.
   - Case 2 : Again by fixing two elements as i and $-i$, the remaining two elements can't be 1 and -1 at the same time. So, in this case, there are 14 different sets, distinct from sets obtained in case 1 and here also $|A-A|\geq|A+A|$; therefore, no MSTD sets.
   - Case 3 : Now if we choose two elements as j and $-j$, then the remaining two elements can't be $\{1,-1\}$ or $\{i,-i\}$ at the same time, thus there will be 13 different sets in this case, which are distinct from sets obtained in case 1 and case 2 and here also $|A-A|\geq|A+A|$.
   - Case 4: Similarly, by fixing two elements as k and $-k$, there are 12 different sets distinct from the sets found in cases 1, 2, and 3. Here also $|A-A|\geq|A+A|$.
   - Case 5 : If A is such that none of $a_i s$ are inverse of each other then $|A-A|=7$ and $|A+A|=8$ and these type of sets are total 16 in number. Therefore we have total of 16 subsets of Q_8, which are MSTD.

(5) If $|A|\geq 5$, then it is a balanced set. Let $A\subseteq Q_8$ and if $\{i,-i\}\notin A$ then if $\{j,-j\}\in A$ then either k or $-k$ has to be in A which will generate whole set Q_8 and if $\{k,-k\}\in A$ then either j or $-j$ has to be in A which will generate whole set Q_8. The same holds when $\{j,-j\}\notin A$ and $\{k,-k\}\notin A$.

Proposition 3.3. *Let $n\geq 2$ be an integer. Let $A_{4n,2}$ denote the collection of subsets of Q_{4n} of size 2. Then $A_{4n,2}$ has strictly more MSTD sets than MDTS sets.*

Proof. Let A be a subset of Q_{4n} of cardinality 2. There are three possible cases to consider for A.

Case 1 : If A contains a^i and a^j where $0\leq i,j\leq 2n-1$ then $A^{-1}=\{a^{2n-i},a^{2n-j}\}$. Here $A+A=\{a^{2i},a^{i+j},a^{2j}\}$. However, $A-A=\{1,a^{2n+i-j},a^{2n+j-i}\}$. Note that $i\neq j$. Both the sumset and the difference set have 3 elements except in one special case. Suppose that $a^{2i}=a^{2j}$. Then, we have that $a^ia^{-j}=a^{-j}a^i$ which implies $a^{i-j}=a^{j-i}$. Thus, when A contains only a^i and a^j, then A is always balanced.

Case 2 : If A contains $\{a^i,a^jb\}$, then $A^{-1}=\{a^{2n-i},a^{n+j}b\}$.
Here $A+A=\{a^{2i},a^{i+j}b,a^{j-i}b,a^n\}$, and $A-A=\{1,a^{i+j+n}b,a^{i+j}b\}$. When $a^{2i}\neq a^n$ and $a^{i+j}b\neq a^{j-i}b$, A will be MSTD. In the case where $a^{2i}=a^n$ then $A+A=\{a^n,a^{\frac{n}{2}+j}b,a^{j-\frac{n}{2}}b\}$ and $A-A=\{1,a^{\frac{n}{2}+j+n}b,a^{\frac{n}{2}+j}b\}$ and in

other case if $a^{i+j}b = a^{j-i}b$ then $A + A = \{1, a^{n+j}b, a^{j-n}b\}$ and $A - A = \{1, a^j b, a^{n+j}b\}$. In both cases, we find that A is a balanced set.

Case 3 : If $A = \{a^i b, a^j b\}$, for some i and j between 0 and $2n - 1$, then $A^{-1} = \{a^{n+i}b, a^{n+j}b\}$. Here $A + A = \{a^n, a^{n+i-j}, a^{n+j-i}\}$ and $A - A = \{1, a^{i-j}, a^{j-i}\}$, again A is a balanced set.

□

Proposition 3.4. *Let $2 \le n \le 4$, and let $A_{4n,3}$ denote the collection of subsets of Q_{4n} of size 3. Then $A_{4n,3}$ has strictly more MDTS sets than MSTD sets.*

Proof. Let A be subset of Q_{4n} of cardinality 3.

Possibilities for cardinality 3 : $\{a^i, a^j, a^k\}, \{a^i, a^j, a^k b\}, \{a^i, a^j b, a^k b\}$, $\{a^i b, a^j b, a^k b\}$, where $0 \le i, j, k \le 2n - 1$.

Case 1 : If $A = \{a^i, a^j, a^k\}$ where $i \ne j \ne k$, then

$$A + A = \{a^{2i}, a^{i+j}, a^{i+k}, a^{2j}, a^{2k}, a^{j+k}\},$$

and

$$A - A = \{1, a^{i-j}, a^{i-k}, a^{j-i}, a^{j-k}, a^{k-i}, a^{k-j}\}$$

implies that $|A - A| \ge |A + A|$.

Case 2 : If $A = \{a^i, a^j, a^k b\}$ where $i \ne j$, then

$$A + A = \{a^{2i}, a^{i+j}, a^{i+k}b, a^{2j}, a^{j+k}b, a^{k-i}b, a^{k-j}b, a^n\},$$

and

$$A - A = \{1, a^{i-j}, a^{n+i+k}b, a^{j-i}, a^{n+j+k}b, a^{k+i}b, a^{k+j}b\}.$$

A is MSTD precisely when either $a^{i+k}b \ne a^{k-i}b$ or, $a^{j+k}b \ne a^{k-j}b$ or both hold. This could fail to occur if only if $i = 0$ or $i = n$ and for j, this could fail to occur if and only if $j = 0$ or $j = n$. The possibility to get MSTD sets is when $j - i = n$ and $i + j \ne n$. In such cases, we get a few MSTD sets; the rest are either MDTS or balanced sets. As a result, we notice that for $n = 2, 3$, or 4, there are more MDTS sets than MSTD sets. However, for $n \ge 5$, the situation becomes more complex, making it difficult to determine the behavior of the subsets in this case.

Case 3 : If $A = \{a^i, a^j b, a^k b\}$ where $j \ne k$, then

$$A + A = \{a^{2i}, a^{i+j}b, a^{i+k}b, a^{j-i}b, a^n, a^{n+j-k}, a^{k-i}b, a^{n+k-j}\},$$

and

$$A - A = \{1, a^{i+j}b, a^{n+i+k}b, a^{j-k}, a^{n+j+i}b, a^{k-j}, a^{k+i}\}.$$

If $n + j \ne 2i$ or $2i \ne n + k - j$, then we can get a few sets that are MSTD, but these sets are very rare. Most sets will be either balanced or MDTS.

Case 4 : If $A = \{a^i b, a^j b, a^k b\}$ where $i \ne j \ne k$, then

$$A + A = \{a^n, a^{n+i-j}, a^{n+i-k}, a^{n+j-i}, a^{n+j-k}, a^{n+k-i}, a^{n+k-j}\},$$

and

$$A - A = \{1, a^{i-j}, a^{i-k}, a^{j-i}, a^{j-k}, a^{k-i}, a^{k-j}\}.$$

If in any case, $|A + A|$ is reducing that will reduce the $|A + A|$ also, this implies that $|A - A| = |A + A|$.

Therefore, from all the cases above, we can conclude that for $n = 2, 3$, or 4, there are more MDTS sets than MSTD sets. □

For $n \geq 5$, we conjecture that the number of sum-dominant subsets of cardinality three in Q_{4n} is greater than the number of difference-dominant subsets of the same cardinality. Thus, we see that the behavior of sum-dominant sets in quaternion groups is different from dihedral groups.

Lemma 3.5. *Let Q_{4n} be a generalized quaternion group of size $4n$, and let $A \subseteq Q_{4n}$ with $|A| = 2n$, $A = \{1, a, a^2, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b\}$ then A is an MSTD set.*

Proof. Let $A = \{1, a, a^2, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b\}$ then
$A^{-1} = \{1, a^{2n-1}, a^{2n-2}, \ldots, a^{n+1}, a^n b, a^{n+1}b, \ldots, a^{2n-1}b\}$.
Here, $A + A = \{1, a, a^2, a^3, \ldots, a^{2n-1}, b, ab, a^2b, \ldots, a^{2n-1}b\} = Q_{4n} \implies |A + A| = 4n$. To prove: $|A - A| = 4n - 1$, we shall show $a^n \notin A - A$. Suppose $a^n \in A - A$ then there is some i, $0 \leq i \leq 2n - 1$ such that $a^n = a^i b a^i b \in A - A$ but the only common element in A and A^{-1} is 1. Hence $a^n \notin A - A$. ☐

Theorem 3.6. *Q_{4n} has at least 2^{2n} subsets which are MSTD where $n \geq 2$.*

Proof. For $n = 2$, Q_8 has 2^4 subsets, which are MSTD. In Q_{4n} we know that $A = \{1, a, a^2, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b\}$ is an MSTD set by above lemma. Replacing a with its inverse will give us a new set, and that set will also be an MSTD. Every element of A can be replaced by its inverses, and 1 can be replaced by a^n, which will give new MSTD sets. By this process, we get a total of $\binom{2n}{1}$ new MSTD sets. Similarly, we can choose two distinct elements from A in $\binom{2n}{2}$ ways, and by replacing them with their respective inverses and 1 by a^n we get new MSTD sets and we can proceed with choosing more elements up to $2n$. Hence we will get total $\binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \binom{2n}{3} + \cdots + \binom{2n}{2n-1} + \binom{2n}{2n} = 2^{2n}$ different sets which are MSTD. ☐

Theorem 3.7. *Let $A \subseteq Q_{4n}$ and $|A| > 2n$ then $|A + A| = |A - A| = |Q_{4n}|$.*

Proof. Let $g \in Q_{4n}$ and assume it is not in $A + A$.

(1) Take any $a \in A$ then $ga^{-1} \notin A$ since $g = ga^{-1}a$ would be in $A + A$.
(2) The map $\phi : Q_{4n} \longrightarrow Q_{4n}$, $a \mapsto ga^{-1}$ is bijective so we get that $|\phi(A)| = |A| > 2n$.

But by (1), we see that $\phi(A) \cap A = \emptyset$, this is a contradiction since $\phi(A) \cup A$ is a disjoint union in Q_{4n}. So, $|\phi(A) \cup A| > 2n + 2n = 4n$, which is contradiction to $|Q_{4n}| = 4n$. So our assumption that g is not in $A + A$ is false and $Q_{4n} \subset A + A$. Therefore $|Q_{4n}| = |A + A|$. Similarly we can prove that $|Q_{4n}| = |A - A|$. ☐

Note that by Proposition 3.4, we know that MDTS sets are more than MSTD sets in Q_{4n} of size 3 for $n \leq 4$. A similar analysis can be done for subsets of cardinality 4, and one can show that there are more MDTS sets. This leads us to the following conjecture.

Conjecture 3.8. *Let $n \geq 2$ be an integer, and let $A_{Q_{4n},m}$ denote the collection of all subsets of Q_{4n} of size m. For $m \leq 2n$, $A_{Q_{4n},m}$ has at least as many MDTS sets as MSTD sets.*

4. Expected Size Of Sum and Difference Sets

We may write Q_{4n} as

$$(\mathbb{Z}/(2n) \rtimes \mathbb{Z}/(4))/\langle (n, 2) \rangle$$

for $n \geq 2$, where the group law on $(\mathbb{Z}/(2n) \rtimes \mathbb{Z}/(4))$ is given by $(a, b)(c, d) = (a + (-1)^b c, b + d)$ and $(n, 2)$ is in center of $\mathbb{Z}/(2n) \rtimes \mathbb{Z}/(4)$ with order 2. It is called

both the generalized quaternion group of order $4n$ and the dicyclic group of order $4n$. Throughout this section, we use $A_{4n,m}$ to denote the set of subsets of size m in Q_{4n}.
Recently Ascoli et al. in [1] tried to prove that $S_{2n,m}$ has more MSTD sets than MDTS sets for values of m greater in order of magnitude than $\sqrt{n}$ with the help of the method of collision analysis, where $S_{2n,m}$ denotes the set of subsets of size m in D_{2n}. The intuition for this comes from the fact that the sum and difference sets for $A \subseteq D_{2n}$ should very roughly have size to be of the order of magnitude of A^2. Hence, one would expect to usually have $A + A = A - A = D_{2n}$ when m is much greater than $\sqrt{n}$. The analysis for relative numbers of MSTD and MDTS sets in $S_{2n,m}$ for these larger values of m should, therefore be based on counting the number of missed sums and differences in D_{2n}, in direct analogy with the case of slow decay for the integers in [2].
They have taken the first steps toward such an analysis by proving the following special case.

Theorem 4.1 ([1]). *If n is prime, and A is chosen uniformly at random from $S_{2n,m}$, then the expected size of $|A - A|$,*

$$\mathbb{E}[|A-A|] = 2n - \frac{nm2^m\binom{n}{m} + 2n(n-1)\binom{n-m-1}{m-1}}{m\binom{2n}{m}} - \frac{n^2(n-1)}{\binom{2n}{m}} \sum_{k=1}^{m-1} \frac{\binom{n+k-m-1}{m-k-1}\binom{n-k-1}{k-1}}{k(m-k)}.$$

We have used the lemmas given in [1] and applied them to generalized quaternion groups and found the expected value of difference sets in Q_{4n}.

Theorem 4.2. *If n is prime, and A is chosen uniformly at random from $A_{4n,m}$, then*

$$\mathbb{E}[|A-A|] = 4n - \frac{n2^{m+1}\binom{2n}{m}}{\binom{4n}{m}} - \frac{4n^4(2n-1)}{\binom{4n}{m}} \sum_{k=0}^{m}\left[\sum_{\substack{(k_1,k_2)\in\mathbb{Z}^2_{\geq 0}\\ k_1+k_2=k}} \frac{\binom{2n-k_1-1}{k_1-1}\binom{n-k_2-1}{k_2-1}}{k_1k_2} \sum_{\substack{(t_1,t_2)\in\mathbb{Z}^2_{\geq 0}\\ t_1+t_2=k}} \frac{\binom{2n+t_1-m-1}{m-t_1-1}\binom{n+t_2-m-1}{m-t_2-1}}{(m-t_1)(m-t_2)}\right].$$

Proof. The proof is similar to that of Theorem 4.1, given in [1]. □

5. Conclusion

We have shown that Generalised Quaternion groups behave differently than Generalised Dihedral Groups, though they have their highly related presentations. A natural question to ask is what are the total number of MSTD or MDTS sets in Q_{4n}. An immediate future direction of research is to prove the Conjecture 3.8. A possible approach to prove this conjecture is to construct an injective map from MDTS sets to MSTD sets in the group. Such an approach has proven to be difficult but has the potential advantage of working for both large and small values of m.

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References

[1] R. Ascoli, J. Cheigh, G. Zeus Dantas e Moura, R. Jeong, A. Keisling, A. Lilly, S. Miller, P. Ngamlamai and M. Phang, *Sum and difference sets in generalized dihedral groups*, ArXiv E-prints. 10 (2022), arXiv.2210.00669.

[2] P. Hegarty, S. Miller, *When almost all sets are difference dominated*, Random Structures and Algorithms. 35 (2009), 118-136.

[3] P. Hegarty, *Some explicit constructions of sets with more sums than differences*, Acta Arithmetica. 130 (2007), 61-77.

[4] G. Martin, K. O'Bryant, *Many sets have more sums than differences*, in: Additive Combinatorics, CRM Proc. Lecture Notes 43 (2006), 287-305.

[5] S. Miller, K. Vissuet, *Most subsets are balanced in finite groups*, Combinatorial and Additive Number Theory. Springer Proceedings in Mathematics and Statistics. 101 (2014), 147-157.

[6] M. B. Nathanson, *Sets with more sums than differences*, J. Integers Seq. 7 (2007).

[7] M. B. Nathanson, *Problems in additive number theory, I*, in: Additive Combinatorics, CRM Proc. Lecture Notes, Amer.Math. Soc. 43 (2007), 263-270.

[8] N. F. Rahin, N. H. Sarmin, and S. Ilangovan, *The non-normal subgroup graph for generalized quaternion Groups*, Proceedings of Science and Mathematics. 7(2022), 20-24.

[9] Y. Zhao, *Counting MSTD sets in finite abelian groups*, Journal of Number Theory. 130 (2010), 2308-2322.

[10] Y. Zhao, *Sets characterized by missing sums and differences*, Journal of Number Theory. 131 (2011), 2107-2134.

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